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PHYSICS

COMPUTATION OF THE TRANSPORT COEFFICIENTS USING THE SHIELDED COULOMB POTENTIAL

Richard L. Liboff

June 20, 1958

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Institute of Mathematical Sciences

NEW YORK UNIVERSITY
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COMPUTATION OF THE TRANSPORT COEFFICIENTS USING THE SHIELDED COULOMB POTENTIAL

by

Richard L. Liboff

ABSTRACT

In the classical Chapman-Enskog theory of a fully ionized gas, the transport coefficients evolve as functions of certain weighted total scattering cross-sections. If a Coulomb potential is assumed between scattering particles, these scattering integrals diverge. A common procedure to circumvent this difficulty is to integrate the Coulomb results up to the Debye cut-off. A more appropriate procedure involves the use of the shielded Coulomb potential, in which case the scattering integrals converge, and there is no need to introduce a cut-off. These integrals are calculated using the shielded Coulomb potential and the results are compared with those of the standard "Coulomb plus Debye cut-off" calculations. To the "dominant log term" the two procedures give identical results. To terms of the next order of accuracy the difference is seen to remain small.

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1. Introduction

In the classical method of Chapman and Enskog[1] for the evaluation of the transport coefficients of a fully ionized gas one starts with the Boltzmann equation. Through an expansion of the distribution function together with solutions to the subsequent linear integral equations approximations to the transport coefficients are generated.

These approximations to the transport coefficients entail the evaluation of certain weighted total scattering cross-sections. If the interaction potential between the ions is assumed to be the Coulomb potential, then these weighted cross-sections diverge. The divergence of such cross-section integrals is well known in scattering theory, and is due to the long range nature of the Coulomb force.

In order to circumvent these difficulties, a common procedure is to introduce a cut-off parameter in the limits of the integrals instead of allowing them to range over all possible distances between the interacting particles. The result is finite. However, the choice of the cut-off value is rather arbitrary since it is based solely on physical arguments which are dictated by the nature of the particular problem.

For example, Marshak,[2] in calculating these terms for star gases, uses a quantum mechanical argument to obtain a cut-off value, indicating that the uncertainty principle would prohibit the use of a classical scattering evaluation for scattering angles beneath some minimum angle, which he chooses as the cut-off[1]. In such a calculation one fails to account for ions scattered through this "diffraction" region.

---

[1] Cut-off in angle and in impact parameter shall be used interchangeably, due to the identity, $s\,ds = \sigma(\cos\theta)\,d\cos\theta$, where s = impact parameter, σ = cross-section, θ = angle of scatter.

Landshoff[3] uses a similar argument in determining the cut-off; however, the uncertainty principle in this instance is used to discard "head-on" collisions, so that the scatterings through angles, $\theta \approx \pi$ are excluded, (opposite to Marshak's use of the uncertainty principle, which precludes scatterings through $\theta \approx 0$). To remove the more troublesome scatterings, $\theta \approx 0$, Landshoff exludes collisions which last longer than a time necessary for the ionized gas to offer a "shielding effect."

Other workers in the field have suggested that the inter-ionic distance should be chosen as the cut-off while still another choice is the "Debye shielding distance."[4] Roughly speaking this distance is the distance within an ionized gas at which the thermal energy is balanced by the electrostatic energy.[2]

The shielding effect was first mentioned in the theory of electrolytes by Debye and Hückel[6]. They showed that the interaction potential between the ions in an electrolyte is not the Coulomb potential $\sim \frac{1}{r}$, but rather a shielded Coulomb potential of the form, $\sim \frac{1}{r} e^{-r/h}$, where h is the effective shielding distance, commonly referred to as the "Debye shielding distance."

More recently it has been shown that a similar argument holds for an ionized gas. If one assumes such a modified Coulomb potential rather than the Coulomb potential in the above method for calculation of the transport coefficients, the related divergences in the weighted total cross-sections disappear. For those temperatures and densities of interest, as is shown later, the Debye distance is much greater than the inter-particle distance, which implies that a binary collision approach is not appropriate. However, it can be shown that calculations, resulting from the more appropriate Fokker-Planck equation (derived independently of the Boltzmann equation) agree with those of the Boltzmann equation to the "dominant log term."[7],[8]

---

[2] It is interesting to note that Langmuir[5] has defined a plasma to be an ionized gas for which the Debye length is small compared to other macroscopic lengths of interest.

We shall concern ourselves in this paper with justifying the usual procedure which assumes a Coulomb potential together with a Debye cut-off in the evaluation of the transport coefficients. To this end, the relevant coefficients evaluated using the conventional cut-off, will be compared with those which are obtained by assuming a shielded Coulomb potential. This should be considered a first step in the direction of obtaining consistent values of the transport coefficients since the assumption of a spherically symmetric shielding cloud is only an approximation to a more complicated form due to inertial causes.[9] Secondly the adopted potential is assumed to remain unmodified during collisions.

The author wishes to acknowledge his grateful appreciation to Harold Grad and Marian H. Rose for their valuable assistance in preparing this report.

2. The Transport Coefficients

The following formulas for the various transport coefficients have been derived in the usual Chapman Enskog manner.[1]

For a two fluid gas composed of ions and electrons, so that $m_e \ll m_i$, it can be shown that the "second approximation" to the coefficient of viscosity μ, reduces to that of a single gas composed of only ions. The related formula appears as

$$\mu = 5kT/8\,\Omega_i^{(2)}(2). \tag{2.1}$$

The interaction between the molecules appears in the calculation of $\Omega_i^{(2)}$, which is defined, in general, by

$$\Omega^{(\ell)}(\gamma) = \sqrt{\pi}\int_0^\infty e^{-V^2}\, V^{2\gamma+2}\Lambda^{(\ell)} dV \tag{2.2}$$

$$\Lambda^{(\ell)} = \int_0^\infty (1-\cos^\ell\theta)\, s\, ds \tag{2.3}$$

where V is a non-dimensional velocity, given by $V^2kT = \frac{1}{2}mv^2$, θ is the angle of scatter, (cf. Appendix 1) and s is the impact parameter.

The viscosity coefficient μ determines the thermal conductivity coefficient λ through the relation $\lambda = 2.7C_v\mu$. C_v is the specific heat at constant volume.

The second approximation to the coefficient of diffusion D_{ie} (subscripts ie indicating diffusion of an ion-electron gas) is written

$$D_{ie} = \frac{3kTM}{16n\,\Omega^{(1)}(1)}$$

where n is the number density of the particles and M is the reduced mass of m_i and m_e.

The related second approximation for the electrical conductivity $\tilde{\theta}$, is given by

$$\tilde{\theta} = \frac{n_i n_e n}{\rho^2 kT} (\varepsilon_i m_e - \varepsilon_e m_i)^2 D_{ie}$$

$$\dashrightarrow \frac{\varepsilon^2 n_e n}{kTn_i} D_{ie}; \quad \text{for } (m_e/m_i) << 1.$$

ρ is the mass density.

3. The Shielded Coulomb Potential

The expression for the transport coefficients given in the preceding section show that the problem is one of evaluating the $\Lambda^{(\ell)}$ integrals. In the introduction we have already stated that these integrals diverge if a Coulomb potential is assumed between particles. However, the introduction of free charge (positive or negative) in an otherwise neutral plasma will induce a charge separation which, in turn, will modify the potential distribution about the charge. This modified potential takes the form of an attenuated Coulomb potential, the attenuation being due to the shielding effect of the induced charge distribution.

The equations of equilibrium with one fixed point charge in an ionized gas are written (esu)

$$(3.1) \qquad \nabla p_r = q_r E$$

$$(3.2) \qquad \nabla \cdot E = 4\pi(q + \varepsilon_1 \delta(r))$$

$$(3.3) \qquad p_r = \frac{q_r}{\varepsilon_r} kT_r \; .$$

E is the electric field vector. The subscript r denotes the r-th component of the gas (e.g. for an ion-electron gas r = i and e). $\delta(r)$ is the Dirac delta function. q is seen to be the "induced" charge distribution which is generated by the fixed point charge ε_1.

Taking the gradient of (3.3) gives:

$$(3.4) \qquad \nabla p_r = \frac{kT_r}{\varepsilon_r} \nabla q_r \; .$$

(T_r is assumed constant).

Substituting (3.1) in (3.4) gives

$$(3.5) \qquad \nabla q_r = \frac{\varepsilon_r q_r}{kT_r} E .$$

Summation over r yields:

$$(3.6) \qquad \sum \nabla q_r = \nabla q = E \sum \frac{\varepsilon_r q_r}{kT_r}$$

of which the divergence is

$$(3.7) \qquad \nabla^2 q = (\nabla \cdot E) \sum \frac{\varepsilon_r q_r}{kT_r} + E \cdot \sum \frac{\varepsilon_r \nabla q_r}{kT_r} .$$

We now assume a linear theory and drop the second term in the above equation. The remaining equation, together with (3.2) yields

$$(3.8) \qquad \nabla^2 q = 4\pi \Big(\sum \frac{\varepsilon_r q_r}{kT_r} \Big) \big(q + \varepsilon_1 \, \delta(r) \big)$$

or

$$(3.9) \qquad \nabla^2 q = \frac{1}{h^2} \big(q + \varepsilon_1 \, \delta(r) \big),$$

$$(3.10) \qquad h^{-2} = \Big(\sum \frac{4\pi \varepsilon_r q_r}{kT_r} \Big) = \sum \frac{4\pi n_r e^2}{kT_r} .$$

The solution is shown to be,*

$$(3.11) \qquad q = - \frac{\varepsilon_1}{4\pi h^2} \frac{e^{-r/h}}{r}$$

which is seen to have the correct normalization, viz. $\int_\infty q \, d\tau = -\varepsilon_1$.

If one introduces a potential function ϕ such that,

$$(3.12) \qquad E = -\nabla \phi ,$$

---

*This follows from the well-known identity $(\nabla^2 - k^2)\frac{e^{-rk}}{r} = -4\pi\delta(r)$.

then the related Poisson's equation appears as,

(3.13) $$\nabla^2 \phi = -4\pi q \text{ .}$$

If q is taken to be the total charge distribution, $\left(\text{i.e.,} \; \varepsilon_1 \delta(r) - \frac{\varepsilon_1 e^{-r/h}}{4\pi k^2 r} \right)$, then the solution to the above equation is

(3.14) $$\phi = \varepsilon_1 \frac{e^{-r/h}}{r}$$

and the corresponding Coulomb potential is

(3.15) $$\phi = \varepsilon_1 / r \text{ .}$$

As already mentioned in the introduction, the length h was first discovered by Debye and Hückel[6] in investigations concerning the effective fields in an electrolyte. The concept was subsequently extended to plasmas by Langmuir, Spitzer and others. The form (eq. 3.10) derived in this section for the Debye distance agrees with the conventional expression.

Equation (3.10) shows that $h \sim \frac{1}{\varepsilon}\sqrt{\frac{T}{n}}$. This is in accord with the interpretation of the Debye distance as being that length at which the electrostatic forces are balanced by the kinetic forces. In an ionized gas, the enormous temperatures present are constantly tending to separate the neutral distribution, while the charges tend to maintain an electrostatic balance. The larger the number density n, the less the mean distance between particles, and consequently the greater the electrostatic forces tending to keep electrical neutrality.

The ensuing calculation now assumes the interaction potential between two scattering particles to be of the modified form. In essence a new Boltzmann equation is postulated in which an "effective" potential is substituted for the usual Coulomb potential between individual particles.

The simple rigid sphere model of a gas gives for the mean free path*, ζ,

$$\zeta \sim \frac{1}{\Delta^2 n} \tag{3.16}$$

where, for an ionized gas, the molecular diameter Δ is defined by the relation,

$$\frac{\varepsilon^2}{\Delta} = kT \tag{3.17}$$

Δ is roughly the mean distance of closest approach.

We now consider the product

$$\Delta \zeta = \frac{\varepsilon^2}{kT\Delta^2 n} = \frac{kT}{n\varepsilon^2} = h^2 \tag{3.18}$$

which shows the Debye distance to be the geometric mean between the particle diameter Δ and the mean free path ζ. If we introduce the two non-dimensional quantities ξ, and Ξ , as,

$$\xi = \Delta/h \tag{3.19}$$

and

$$\Xi = \zeta/h \tag{3.20}$$

then it follows from (3.19) that

$$\xi \, \Xi = 1 . \tag{3.21}$$

---

*For a plasma, it is more correct to take the mean free path to be the above value divided by log ξ(cf. eq. 20), also [5].

4. The $\Lambda^{(\ell)}$ integrals

As has already been stated in part 2 of this report, the calculation of the transport coefficients involves principally, knowledge of the $\Lambda^{(\ell)}$ integrals. In calculating these integrals, it is convenient to introduce a length L, intermediate between the particle diameter Δ and the Debye length h, i.e.,

$$\Delta << L << h . \tag{4.1}$$

The length L has the following properties: those interactions which occur within a sphere of radius L are governed by a Coulomb potential. For interactions with impact parameter greater than L, only grazing collisions are important, and further these interactions are governed by a screened Coulomb field. In essence, we are assuming a new form for the potential, namely,

$$\phi \sim \frac{1}{r} \qquad r < L \tag{4.2}$$

and

$$\phi \sim \frac{\exp(r/h)}{r} \qquad r > L \tag{4.3}$$

which is seen to be an accurate representation of the modified Coulomb potential for proper values of L.

Correspondingly in the $\Lambda^{(\ell)}$ integrals, which are defined by,

$$\Lambda^{(\ell)} = \int_0^\infty sds(1 - \cos^2\theta), \tag{4.4}$$

the integration is divided into two regions, so that,

$$\Lambda^{(\ell)} = \int_0^L + \int_L^\infty . \tag{4.5}$$

In the first integral the exact Coulomb expression is used for the scattering angle θ; in the "far region" ($s > L$) the form for θ corresponding to the potential (4.3) is employed together with the assumption that in this 'far' region the high energy, small angle scatterings are dominant.

As is well known, the $\Lambda^{(\ell)}$ integrals diverge logarithmically when the Coulomb potential is assumed throughout. However if the modified Coulomb potential is used, the integrations converge.

Before evaluating $\Lambda^{(2)}$ we introduce two non-dimensional parameters α, and σ, through the equations,

$$(4.6) \qquad \alpha = \left(\frac{\varepsilon^2}{h}\right)\Big/\frac{mv^2}{2}\ ,$$

and

$$(4.7) \qquad \sigma = s/h,$$

whence we see that small α is indicative of high energy scattering. σ is merely the impact parameter measured in units of the Debye length h.

In the expansion of the first integral of (4.5) in powers of ξ, the leading term appears as $(\alpha h)^2 \log(L/\Delta)$. The corresponding leading term in the expansion of the second integral appears as $(\alpha h)^2 \log(h/L)$. Addition of the two series immediately indicates that the dominant log term is independent of the choice of separation parameter L. $\left((\alpha h)^2\log(L/\Delta) + (\alpha h)^2\log(h/L) = (\alpha h)^2\log(1/\xi)\right)$ The next order of accuracy is also shown to be independent of L.

The angle of scatter θ, for the Coulomb interaction potential is given by the equation,

$$(4.8) \qquad \csc^2(\tfrac{\theta}{2}) = 1 + (2\sigma/a)^2.$$

In Appendix I, θ is calculated for the modified Coulomb potential and the relation

$$(4.9) \qquad \theta = a\, K_1(\sigma)$$

is derived. K_1 is the modified Bessel function of the second kind, first order. If the Coulomb angle of scatter (eq. 4.8) is expanded about $a = 0$, one obtains for related small angles of scatter, $\theta \sim a/\sigma$. Likewise if K_1 is developed in eq. (4.9) it is seen that $\theta \sim a/\sigma$ for $\sigma \sim 0$, so that the result for the modified Coulomb potential reduces to the Coulomb result for small σ. In other words, far within the shielding cloud ($\sigma = \frac{s}{h} << 1$), the shielded potential reduces to the pure Coulomb potential.

We are now prepared to calculate the $\Lambda^{(\ell)}$ integrals. As was seen in part II, it is sufficient to consider the problem for $\ell = 1$, and $\ell = 2$ only, as these are the quantities involved in the second approximation to all of the transport coefficients. The quantity $\Lambda^{(2)}$ is written

$$(4.10) \qquad \Lambda^{(2)} = \int_0^\infty [\sin^2\theta] s\,ds = h^2 \int_0^\infty [\sin^2\theta]\sigma\,d\sigma$$

or in accordance with (4.5)

$$(4.11) \qquad \Lambda^{(2)} = h^2 \int_0^{\mathcal{L}} (\sin^2\theta_c)\sigma\,d\sigma + h^2 \int_{\mathcal{L}}^\infty (\sin^2\theta_{c'})\sigma\,d\sigma$$

$\mathcal{L}$ is L measured in units of Debye length.

The subscripts c and c' refer respectively to the Coulomb and shielded Coulomb formulation. Substitution of (4.8) and (4.9) for the angles Θ_c and $\Theta_{c'}$ gives for $\Lambda^{(2)}$,

$$(4.12) \qquad \Lambda^{(2)} = \left(\frac{4h}{a}\right)^2 \int_0^{\mathcal{L}} \frac{\sigma^3 d\sigma}{\left(1+(2\sigma/a)^2\right)^2} + (ah)^2 \int_{\mathcal{L}}^{\infty} K_1^2(\sigma)\sigma d\sigma$$

which upon integration yields,

$$(4.13) \qquad \Lambda^{(2)} = \frac{(ah)^2}{2} \left\{ \log\left(1+(2\mathcal{L}/a)^2\right) - \frac{(2\mathcal{L}/a)^2}{1+(2\mathcal{L}/a)^2} + 2\mathcal{L}K_1K_0 + (\mathcal{L}K_0)^2 - (\mathcal{L}K_1)^2 \right\}.$$

The Bessel functions K_0 and K_1 are evaluated at $\sigma = \mathcal{L}$. By definition, $\mathcal{L} << 1$, and furthermore we shall restrict the calculations to the region $(2\mathcal{L}/a)^2 >> 1$ (cf. part V for regions of application). Using these simplifications in the expansions of the Bessel functions and log term in (4.13) $\Lambda^{(2)}$ becomes,

$$(4.14) \qquad \Lambda^{(2)} = (ah)^2 \left\{ \ln\left(\frac{2}{a}\right) - 1 - \gamma + \ln 2 \right\}.$$

This is to be compared with the expression obtained by integrating the pure Coulomb formulation up to the Debye distance ($\mathcal{L} = 1$). Calling this form $\Lambda_c^{(2)}$, we have,

$$(4.15) \qquad \Lambda_c^{(2)} = (ah)^2 \left\{ \ln\left(\frac{2}{a}\right) - 1 \right\}.$$

These last two equations show that to within the log term the Coulomb-with-cut-off and screened Coulomb formulations give identical results for the $\Lambda^{(2)}$ integrals.

Recalling the defining equations for a (cf. eq. 4.6) and the variable $\xi = \Delta/h$ (cf. eq. 3.19), it is easily shown that α may be rewritten:

$$\alpha = \xi/2V^2 \tag{4.16}$$

where V is the non-dimensionalized velocity $V^2 = \frac{mv^2}{2}/kT$.

This expression for α allows eq's. (4.14) and (4.15) to be written in the more familiar forms

$$\Lambda^{(2)} = (\alpha h)^2 \left\{ \ln(\tfrac{h}{\Delta}) + 2\ln(2V) - .884 \right\} \tag{4.17}$$

$$\Lambda_c^{(2)} = (\alpha h)^2 \left\{ \ln(\tfrac{h}{\Delta}) + 2\ln(2V) - 1 \right\} . \tag{4.18}$$

$\ell = 1$

From the definition (4.4), the $\Lambda^{(1)}$ integral is

$$\Lambda^{(1)} = \int_0^\infty (1-\cos\theta)\,s\,ds \tag{4.19}$$

or, more conveniently,

$$\Lambda^{(1)} = 2\int_0^\infty \sin^2\frac{\theta}{2}\,s\,ds. \tag{4.20}$$

Using the same argument as for $\Lambda^{(2)}$, the domain of integration of $\Lambda^{(1)}$ is divided into two regions by the length L giving

$$\Lambda^{(1)} = 2\int_0^L \sin^2\frac{\theta}{2}\,s\,ds + 2\int_L^\infty \sin^2\frac{\theta}{2}\,s\,ds. \tag{4.21}$$

In the "far region", s > L, the approximation is again employed that only scatterings of high energy and small angle are relevant, so that $\sin^2 \frac{\theta}{2}$ is replaced by $(\frac{\theta}{2})^2$. Thus the second integral in the above form becomes $\frac{1}{2}\int_L^\infty \theta^2 s ds$, or one-half the related equivalent part of the $\Lambda^{(2)}$ integral.

For the first integral, we have from the Coulomb formulation

$$(4.22) \qquad \sin^2 \frac{\theta}{2} = \frac{1}{1 + (2\sigma/\alpha)^2}$$

so that

$$(4.23) \qquad 2 h^2 \int_0^{\mathcal{L}} \sin^2 \frac{\theta}{2} \, \sigma d\sigma = \frac{1}{4}(\alpha h)^2 \ln\left(1 + (\frac{2\mathcal{L}}{\alpha})^2\right) .$$

This gives the well-known result that to within the log term in the expansion of the $\Lambda_c^{(2)}$ and $\Lambda_c^{(1)}$ integrals we have

$$(4.24) \qquad \Lambda_c^{(1)} = \frac{1}{2}\Lambda_c^{(2)} .$$

However, if all terms of O(1) are included, then the modified Coulomb formulation yields

$$(4.25) \qquad \Lambda^{(1)} = \frac{(\alpha h)^2}{2} \left\{ \ln(\frac{2}{\alpha}) - \frac{1}{2} - \gamma + \ln 2 \right\} ,$$

whereas the ordinary Coulomb formulation with Debye cut-off ($\mathcal{L} = 1$) yields

$$(4.26) \qquad \Lambda_c^{(1)} = \frac{(\alpha h)^2}{2} \left\{ \ln \frac{2}{\alpha} \right\} .$$

In terms of the Debye distance h, and the ionic diameter Δ, these latter two results appear as,

$$\Lambda^{(1)} = \frac{(\alpha h)^2}{2} \left\{ \ln\left(\frac{h}{\Delta}\right) + 2 \ln(2V) - \frac{1}{2} - \gamma + \ln 2 \right\} \tag{4.27}$$

$$\Lambda_c^{(1)} = \frac{(\alpha h)^2}{2} \left\{ \ln\left(\frac{h}{\Delta}\right) + 2 \ln(2V) \right\}. \tag{4.28}$$

For convenience we recall the parallel results obtained for $\Lambda^{(2)}$ viz.,

$$\Lambda^{(2)} = (\alpha h)^2 \left\{ \ln\left(\frac{h}{\Delta}\right) + 2 \ln(2V) - 1 - \gamma + \ln 2 \right\} \tag{4.29}$$

$$\Lambda_c^{(2)} = (\alpha h)^2 \left\{ \ln\left(\frac{h}{\Delta}\right) + 2 \ln(2V) - 1 \right\} \tag{4.30}$$

(We note that $-\gamma + \ln 2 = .116$).

5. Conclusions

A. Regions of application

The major restriction imposed, in deriving eq. (4.17) of the previous section, was that the quantity $(2\mathcal{L}/\alpha)^2$ was to be large compared to unity, and that $\mathcal{L}$ was to be small compared to unity.

From the relation $\alpha = \xi/2V^2$, $2\mathcal{L}/\alpha$ may be rewritten $4\mathcal{L}V^2/\xi$. The quantity ξ is the "particle diameter" measured in units of the Debye distance. Expressed in terms of the charge ε, the number density n, and the temperature T, ξ is written,

$$(5.1) \qquad \xi = \sqrt{n}\left(\frac{\varepsilon^2}{kT}\right)^{3/2} .$$

The constant ratio $\varepsilon^3/k^{3/2}$ has the numerical value $\sim 10^{-6}$ (in esu), so that the condition $(2\mathcal{L}/\alpha)^2 \gg 1$, or $(4\mathcal{L}V^2/\xi)^2 \gg 1$, may be rewritten, $\left(\frac{T^{3/2}4\mathcal{L}V^2}{n^{1/2}10^{-6}}\right)^2 \gg 1$.

The value of $\mathcal{L}$ is chosen by realizing that $e^{-\mathcal{L}}/r$ represents $1/r$ to within an error of 1 percent if $\mathcal{L}$ is 10^{-2}. The non-dimensional velocity V is taken to be $\sim O(1)$. We may now write the above condition in the form,

$$(5.2) \qquad T^3/n \gg 10^{-8}(^\circ K - cm)^3.$$

From this relation, the present formulation is seen to hold for a wide variety of temperatures and densities of interest.

B. Results and remaining problems

We have seen that when the Coulomb potential is assumed to be the interaction potential between particles of a gas, a "cut-off procedure" is necessitated in the calculation of the transport coefficients. The cut-off parameter is introduced in the evaluation of the $\Lambda^{(\ell)}$ integrals. These integrals are similar to the total cross-section integrals of scattering theory. They differ by the weighting factor $(1 - \cos^{\ell}\theta)$. Such integrals normally converge for inverse power potentials of the form r^{-n}, for all values of n greater than 1. In the Coulomb case n = 1, the $\Lambda^{(\ell)}$ integrals (for all ℓ) are seen to diverge logarithmically.

When an extraneous charge is introduced into a neutral plasma a shielding cloud is generated about it which attenuates the original Coulomb field exponentially. If this "modified Coulomb" potential is used in the $\Lambda^{(\ell)}$ integrals, they of course converge.

In the conventional calculation of the transport coefficients of an ionized gas, the Coulomb potential, together with the Debye cut-off is used. We have compared the results so obtained with the transport coefficients evaluated using the more relevant shielded Coulomb potential.

The results show that to the "dominant log term" there is no difference between the two calculations. To terms of higher accuracy, for the diffusion coefficient and electrical conductivity, the related error is of the order of two percent, and for the viscosity and thermal conductivity the error is of the order of one tenth of one percent.

The present results have been derived on the assumption of a spherically symmetric form for the modified potential. As has already been mentioned, this assumption must be altered, for due to the motion of both scattering particles, there will be an inertial change in the shape of the screening cloud.[9] In addition the modification of the charge distribution during collisions must also be accounted for.

In the light of these changes, it is seen that what has been presented so far is a first approximation to a more general and accurate description of the mechanisms involved.

APPENDIX I. Classical Scattering Formulation for a Screened Coulomb Potential

Classically, (when the orbit is well defined) the angle of scatter for an arbitrary potential ϕ is given by[10]

$$(A1.1)\qquad \theta = \pi + 2\int_0^{u_0} \frac{s\,du}{\sqrt{1-(\phi(u)/E)-(su)^2}}\ ,$$

where E is the energy of interaction, s is the impact parameter, u is the inverse radius between the two scattering particles, and u_0 corresponds to the distance of closest approach, which coincides with the root of the form in the above radical.

Defining the three non-dimensional terms, f(u), α, and x, through the equations,

$$(A1.2)\qquad \phi = \phi_0\, f(u),$$

$$(A1.3)\qquad \alpha = \phi_0/E,$$

and

$$(A1.4)\qquad x = u/u_0,$$

eq. (A1.1) becomes

$$(A1.5)\qquad \theta = \pi + 2\int_0^{1} \frac{dx}{\sqrt{[(1-\alpha f(u_0x))/(su_0)^2]-x^2}}\ .$$

The equation determining u_0 is written,

$$(A1.6)\qquad 1 - \alpha f(u_0) - (su_0)^2 = 0,$$

of which a first iteration about $\alpha = 0$ gives

(A1.7) $$1 - \alpha f(1/s) = (su_o)^2.$$

Substituting into (A1.5) the form for u_o indicated by (A1.7) and expanding the resulting equation for θ about $\alpha = 0$ up to linear terms in α gives for θ*

(A1.8) $$\theta = \alpha \int_0^1 \frac{[f(u_o x) - f(1/s)]dx}{(1-x^2)^{3/2}} .$$

Within the order of accuracy of this calculation it is consistent to replace u_o in $f(u_o x)$, in the last formula, by $(1/s)$. We may now write for θ

(A1.9) $$\theta = \alpha \int_0^1 \frac{[f(x/s) - f(1/s)]dx}{(1-x^2)^{3/2}} .$$

Another form of (A1.9) which is helpful in the integration to follow is

(A1.10) $$\theta = -\alpha \int_0^1 \frac{f(1/s)dx}{\sqrt{1-x^2}} + \alpha \int_0^1 \frac{[f(x/s) - x^2 f(1/s)]dx}{(1-x^2)^{3/2}} ,$$

(A1.11) $$= -\alpha(\pi/2)f(1/s) + \alpha \int_0^1 \frac{[f(x/s) - x^2 f(1/s)]dx}{(1-x^2)^{3/2}}$$

The shielded Coulomb potential is of the form

(A1.12) $$\phi = \frac{A \exp(-r/h)}{r/h} ,$$

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*For a similar discussion, cf. Kinetic Theory of Gases, E. H. Kennard, p. 119, McGraw-Hill, 1938.

the non-dimensional part f being the above, less the coefficient A.[†] The function f(x/s) appears as

$$(A1.13) \qquad f = (x/\sigma)\exp(-\sigma/x),$$

$(\sigma h = s)$

which when substituted into (A1.10) gives for θ,

$$(A1.14) \qquad \theta = -\left(\frac{\pi}{2}\right)\left(\frac{\alpha}{\sigma}\right)\exp(-\sigma) + \left(\frac{\alpha}{\sigma}\right)\int_0^1 \frac{\left(x\exp(-\sigma/x) - x^2\exp(-\sigma)\right)}{(1-x^2)^{3/2}}\,dx.$$

Employing the substitution $x = \cos\phi$, followed by a parts integration, yields for (A1.14)

$$(A1.15) \qquad \theta = -\alpha\int_0^{\pi/2} \sec^2\phi\,\exp(-\sigma\sec\phi)\,d\phi.$$

The desired result follows immediately from the integral formula[*]

$$(A1.16) \qquad \int_0^{\pi/2} \sec^n\phi\,\exp(-\sigma\sec\phi)\,d\phi = (-1)^{n-1}K_o^{(n-1)}(\sigma),$$

for the (n-1)st derivative of the zero-order modified Bessel function of the second kind K_o. (Exponents without parentheses are algebraic.) The above formula together with the relation $K_o'(\sigma) = -K_1(\sigma)$ reduces (A1.15) to

$$(A1.17) \qquad \theta = \alpha K_1(\sigma).$$

[†] cf. eq. (3.14).

[*] This formula follows from the relation $K_1(x) = \int_1^{\infty}\frac{e^{-xt}dt}{\sqrt{1-t^2}}$.

The expansion for $K_1(\sigma)$ is

$$(\text{A1.18}) \quad K_1(\sigma) = \left(\frac{1}{\sigma}\right) + \sum_{i=0}^{\infty} \frac{(\sigma/2)^{2i+1}}{i!(i+1)!}\left\{\log(\sigma/2)-(1/2)[\psi(i+1)+\psi(i+2)]\right\},$$

where the ψ functions are given by

$$(\text{A1.19}) \quad \psi(i+1) = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{i}\right) - \gamma, \ \psi(1) = -\gamma.$$

APPENDIX II. Expansions involved in the $\Lambda^{(2)}$ integral

The $\Lambda^{(2)}$ integral was written

$$(A2.1)\qquad \Lambda^{(2)} = \int_0^\infty (1 - \cos^2\theta)\,s\,ds \;=\; \int_0^\infty \sin^2\theta\; s\,ds\;.$$

The integral was approximated by

$$(A2.2)\qquad \Lambda^{(2)} = \int_0^L \sin^2\theta_c\, s\,ds + \int_L^\infty \theta_{c'}^2\; s\,ds.$$

In the first integral, the Coulomb formulation for $\sin^2\theta_c$ gives

$$(A2.3)\qquad \int_0^L \sin^2\theta_c\, s\,ds = \frac{(ah)^2}{2}\left[\log\left(1+\left(\frac{2\mathcal{L}}{a}\right)^2\right) - \frac{\left(\frac{2\mathcal{L}}{a}\right)^2}{1+\left(\frac{2\mathcal{L}}{a}\right)^2}\right],$$

and, to within the relevant approximation $(2\mathcal{L}/a)^2 >> 1$, this part of the integration is approximated by

$$(A2.4)\qquad \int_0^L \sin^2\theta_c\, s\,ds = \frac{(ah)^2}{2}\left[\,2\,\log\left(\frac{2\mathcal{L}}{a}\right) - 1\right]$$

The remaining integration over $\theta_{c'}^2$ is written

$$(A2.5)\qquad \int_L^\infty \theta_{c'}^2\; s\,ds = (ah)^2 \int_{\mathcal{L}}^\infty K_1^2(\sigma)\,\sigma\, d\sigma\;.$$

This integral is expanded as

$$(A2.6)\qquad \int_{\mathcal{L}}^\infty \sigma K_1^2(\sigma)\,d\sigma \;=\; \tfrac{1}{2}\mathcal{L}^2\left\{K_1'^{\,2}(\mathcal{L}) - \left(1+\frac{1}{\mathcal{L}^2}\right)K_1^2(\mathcal{L})\right\}$$

Using the identity

$$(A2.7)\qquad \mathcal{L}^2 K_1'^{\,2} \;=\; (K_1 + \mathcal{L}K_0)^2,$$

allows the above integral to be rewritten

$$(A2.8) \qquad \int_{\mathcal{L}}^{\infty} \sigma K_1^2 d\sigma = \frac{1}{2}\left\{-\mathcal{L}^2K_1^2 + 2\mathcal{L}K_1K_o + \mathcal{L}^2K_o^2\right\}.$$

Expansion of the three terms in (A2.8) yields,

$$(A2.9) \qquad \mathcal{L}^2K_1^2 = (1 + \frac{\mathcal{L}^2}{2} \ln \frac{\mathcal{L}}{2} + \ldots)^2 = 1 + O(\mathcal{L}^2 \ln \frac{\mathcal{L}}{2}),$$

$$(A2.10) \qquad \mathcal{L}^2K_o^2 = (\mathcal{L}^2\gamma + \mathcal{L}^2 \ln \frac{\mathcal{L}}{2} + \ldots)^2 = O(\mathcal{L}^4 \ln \frac{\mathcal{L}}{2}),$$

and for $\mathcal{L}K_1K_o$,

$$(A2.11) \qquad \mathcal{L}K_1K_o = -(\mathcal{L}\gamma + \mathcal{L}\ln \frac{\mathcal{L}}{2} + \ldots)(\frac{1}{\mathcal{L}} + \frac{\mathcal{L}}{2} \ln(\frac{\mathcal{L}}{2}) + \ldots)$$

$$= -(\gamma + \ln \frac{\mathcal{L}}{2}) + O(\mathcal{L}^2 \ln \frac{\mathcal{L}}{2}).$$

These expansions permit the integral (A2.8) to be rewritten as

$$(A2.12) \qquad \int_{\mathcal{L}}^{\infty} \sigma K_1^2 d\sigma = -\frac{1}{2} - \gamma - \ln \frac{\mathcal{L}}{2} + O(\mathcal{L}^2 \ln \frac{\mathcal{L}}{2})$$

Combining the two integrals in (A2.2) gives for $\Lambda^{(2)}$,

$$(A2.13) \qquad \Lambda^{(2)} = (ah)^2 \left\{\ln(\frac{2\mathcal{L}}{a}) - \frac{1}{2} - \frac{1}{2} - \gamma - \ln(\frac{\mathcal{L}}{2}) + O(\mathcal{L}^2 \ln \frac{\mathcal{L}}{2})\right\}$$

Retaining only terms within the brackets of the order of unity, yields finally for $\Lambda^{(2)}$,

$$= (ah)^2 \left\{\ln(\frac{2}{a}) - 1 - \gamma + \ln 2\right\}$$

$$= (ah)^2 \left\{\ln(\frac{2}{a}) - .884\right\}.$$

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On page 11, printed line 4, $\frac{\varepsilon_1 e^{-r/h}}{4\pi k^2 r}$ should read $\frac{\varepsilon_1 e^{-r/h}}{4\pi h^2 r}$.